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COMPARISON BETWEEN THE ABSOLUTE NODAL COORDINATE FORMULATION AND INCREMENTAL PROCEDURES

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ABSTRACT

Many flexible multibody applications are characterized by high inertia forces and motion discontinuities. Because of these characteristics, problems can be encountered when large displacement finite element formulations are used in the simulation of flexible multibody systems. In this investigation, the performance of two different large displacement finite element formulations in the analysis of flexible multibody systems is investigated. These are the incremental corotational procedure proposed by Rankin and Brogan [15] and the non-incremental absolute nodal coordinate formulation recently proposed [19]. It is demonstrated in this investigation that the limitation resulting from the use of the nodal rotations in the incremental corotational procedure can lead to simulation problems even when very simple flexible multibody applications are considered. The absolute nodal coordinate formulation, on the other hand, does not employ infinitesimal or finite rotation coordinates and leads to a constant mass matrix. Despite the fact that the absolute nodal coordinate formulation leads to a complex expression for the elastic forces, the results presented in this study, surprisingly, demonstrate that such a formulation is efficient in static problems as compared to the incremental corotational procedure. The excellent performance of the absolute nodal coordinate formulation in static and dynamic problems can be attributed to the fact that such a formulation does not employ rotations and leads to exact representation of the rigid body motion of the finite element.

1 INTRODUCTION

The computational issues associated with large displacement problems [2, 4, 6, 10, 16, 20] become important when flexible multibody applications are considered. This is due to the high nonlinearities in the equations of motion, the coupling of the elastic and reference motions, high inertia forces and possible motion discontinuities. Therefore, it is important to carefully examine the accuracy, robustness and efficiency of the computational procedures used in the large displacements of multibody system applications. The most widely used formulation in flexible multibody dynamics is the *floating frame of reference formulation* [12, 13, 14, 19]. The use of this formulation, however, has been limited to small deformation problems. In the *floating frame of reference formulation*, the body elastic deformation is described in a body coordinate system, and it is assumed that this deformation is small in order to justify the use of linear modes [19]. Consequently, this formulation has been rarely used for large deformation problems, which can be analyzed more accurately using a full finite element representation. Nonetheless, large deformation problems can be examined using the floating frame of reference formulation by dividing the flexible body into a large number of bodies, each of which has its own body reference. This approach, however, leads to large dimensionality and nonlinearity of the inertia forces.

There are two types of finite element procedures that can be used for the large deformation analysis; incremental and non-incremental. The incremental approach is the most widely used procedure for the solution of non-linear large rotations and large deformation problems in structural applications. Several incremental procedures have been developed for non-isoparametric elements, in which infinitesimal rotations are used as nodal coordinates. In principle, these procedures can also be used in multibody applications, since such procedures can also be used to describe the large reference displacements and rotations which are characteristics of multibody problems. However, the limitations on the rotation increments in some of these procedures, as will be discussed in this paper, and the fact that these pro-

cedures do not lead to an exact rigid body inertia representation as the result of the early linearization of the equations of motion [17], make the incremental approach less attractive to use in flexible multibody problems. Belytschko and Hsieh [6] used the *convected coordinate system* and applied it to the dynamic analysis of structural systems that undergo large rotations. In this formulation, a convected coordinate system is assigned to each finite element and the element internal forces are first defined in the element convected coordinate system and then transformed to the global coordinate system. A basic feature of this technique is the decomposition of the global displacement field into rigid-body and strain-producing deformation components. The increment steps are chosen such that the element rotation between two consecutive configurations is small and the element shape function and local nodal coordinates can be used to describe this small rotation. Argyris et al. [2] presented a detailed discussion on the convected coordinate procedure and the large deflection problems. They introduced the *natural approach* that refers to the separation between the rigid body displacement field and the natural deformation in the total displacement field of a finite element. Hughes and Winget [10] presented an efficient algorithm to define the displacement increment over the step in the large deformation analysis and demonstrated that a unique large rotation vector can be assigned to any rotation in large deformation problems. A corotational procedure for the solution of nonlinear finite element large rotation problems was proposed by Rankin and Brogan [15]. This procedure will be discussed in detail in the following section and will be used in the study presented in this paper. In this procedure, the contribution of the so called large rigid-body rotations of the element is removed from the global displacement field through the use of an element convected coordinate system. A nonsingular large rotation vector is introduced to describe the nodal rotations [3]. Hsiao and Jang [9] extended the use of the corotational procedure to the dynamic analysis of planar flexible linkages. A detailed corotational formulation for the dynamic analysis of planar beams undergoing large deflections has been recently presented by Behdinan et al. [5].

A new non-incremental approach, the absolute nodal coordinate formulation, has recently been proposed [19]. This formulation differs from other existing finite element formulations in the sense that no infinitesimal or finite rotations are used as nodal coordinates. The set of nodal coordinates consists of global displacements and slopes. Using this approach, beams and plates can be treated as isoparametric elements, and therefore there is no need to introduce an element coordinate system to describe the rigid body rotations of the finite element. The absolute nodal coordinate formulation leads to a constant mass matrix, while the elastic forces are nonlinear functions of the element coordinates. In the absolute nodal coordinate formulation, large rigid body displacements including large rotations produce zero strains in the finite elements. It was demonstrated [19] that in order to obtain correct results in the dynamic analysis, a consistent mass approach must be used in this formulation.

It is the objective of this paper to examine the performance of the absolute nodal coordinate formulation by comparing it with the corotational procedure presented by Rankin and Brogan [15] and implemented in the finite element code ANSYS [1]. It will be shown that numerical problems are encountered when the incremental procedure is used in flexible multibody applications. This paper is organized as follows. In Section 2 the incremental corotational procedure proposed by Rankin and Brogan [15] is reviewed. This procedure will be extensively used in our investigation, and therefore, the review materials presented in Section 2 are used as the basis for the discussion presented in the following sections. In Section 3, the non-incremental absolute nodal coordinate formulation is introduced. In Section 4, the performance of the absolute nodal coordinate formulation in static problems is examined and compared with the incremental procedure. Two problems with known analytical solutions are considered. These are the elastica problem and the bending of a beam into a full circle. In Section 5, the main features of the absolute nodal coordinate formulation in the case of dynamics are summarized. Comparison between the incremental corotational procedure and the non-incremental absolute nodal coordinate formulation when flexible multibody appli-

cations are considered is presented in Section 6. Summary and conclusions drawn from this study are presented in Section 7.

2 COROTATIONAL PROCEDURE

The incremental procedure has been widely and successfully used in the nonlinear finite element analysis of large rotation structural problems. The incremental finite element corotational procedure proposed by Rankin and Brogan [15] has been implemented in several general purpose structural analysis codes such as ANSYS, and has been used in the analysis of many large rotation and deformation problems. In this procedure, which is independent of the element formulation, any rigid body motion contribution is eliminated from the global displacement field in order to determine the pure deformation. The contribution of the rigid body rotations of the element is eliminated by using a convected coordinate system that moves with the element. The element equations are first defined in the element coordinate system and then transformed in order to define these equations in the global inertial frame. These equations are solved for the displacement increments that are then used to update the global displacement field of the element.

In this approach, the nonlinear kinematics of the finite element is defined in terms of a large reference motion plus a small deformation; this holds assuming that at each time step the displacement increments are so small that the current configuration in which the element equations are defined can be considered a valid reference configuration. This implies that in one time step there is no large variation in: 1) the deformation within each element, and 2) the large reference motion. Consequently, the most important parameter that governs this procedure is the time/load step which must remain small. It will be shown later that there is another limitation due to the assumption that the total deformation within each element must remain small.

In order to extract the rigid body motion, a local coordinate system is introduced. Using the notation of Rankin and Brogan [15], let $\mathbf{E}_k$ be the orthogonal transformation matrix that defines the orientation of the local element frame in the global frame at the k-th step. A rigid body rotation can be extracted from the total displacement as follows. The portion $\mathbf{u}_e^{def}$ of the total displacement that causes strain is given by

$$\mathbf{u}_e^{def} = \mathbf{E}_k^{\mathrm{T}}(\mathbf{u}_g + \mathbf{X}_g) - \mathbf{X}_e, \tag{1}$$

where $\mathbf{u}_g$ is the total displacement defined in the inertial coordinate system, $\mathbf{X}_g$ is the global position of an arbitrary point, and $\mathbf{X}_e$ is the local position of the same point before deformation. The vector $\mathbf{u}_e^{def}$ will be used to define the strain energy and the generalized elastic forces. The solution of the system of equations at step k yields a displacement increment $\Delta\mathbf{u}_{k+1}$. Using this increment, it is possible to calculate the total displacement $\mathbf{u}_{k+1}$ and use it to define the new orthogonal transformation matrix $\mathbf{E}_{k+1}$. The small rotation increment is used to update the rotations within the element using the corotational approach. In fact, the deformational rotations can be finite rotations, and cannot be treated as ordinary vectors [3]. For this reason, the rotations are described in terms of *pseudovectors* and the nodal deformation rotational degrees of freedom are treated differently from nodal deformation translational degrees of freedom. Rankin and Brogan [15] introduced a *surface* coordinate system rigidly attached to each node. This surface coordinate system is defined in the inertial frame by the orthogonal transformation $\mathbf{S}_k$, as shown in Fig. 1. Clearly the deformation is produced by a relative rotation of $\mathbf{S}_k$ with respect to $\mathbf{E}_k$. This relative rotation is expressed through the orthogonal matrix $\mathbf{T}_k$, where

$$\mathbf{T}_k = \mathbf{E}_k^{\mathrm{T}}\mathbf{S}_k\mathbf{S}_0^{\mathrm{T}}\mathbf{E}_0. \tag{2}$$

Hughes and Winget [10] have shown that the quantity

$$\Omega = 2(\mathbf{T} - \mathbf{I})(\mathbf{T} + \mathbf{I})^{-1} \tag{3}$$

is always skew-symmetric for any orthogonal matrix $\mathbf{T}$. Rankin and Brogan [15] first considered the following definition of $\boldsymbol{\Omega}$, that is slightly different from the one presented by Argyris [3]:

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \frac{2(\mathbf{T}-\mathbf{T}^{\mathrm{T}})}{1+\gamma}, \tag{4}$$

where γ is the trace of $\mathbf{T}$:

$$\gamma = \mathrm{tr}(\mathbf{T}) = 1 + 2\cos\theta, \tag{5}$$

and θ is the angle of rotation about the principal eigenvector of $\mathbf{T}$. The matrix $\boldsymbol{\Omega}$ is associated with the vector

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^{\mathrm{T}}, \tag{6}$$

which is called rotational *pseudovector*, and becomes the rotation vector when rotations are small. Given the *pseudovector* $\boldsymbol{\omega}$, the skew-symmetric matrix $\boldsymbol{\Omega}$ is calculated and used to evaluate the matrix $\mathbf{T}$ as

$$\mathbf{T} = \mathbf{I} + \frac{\boldsymbol{\Omega}+\frac{1}{2}\boldsymbol{\Omega}^2}{1+\frac{1}{4}\left|\boldsymbol{\omega}\right|^2}. \tag{7}$$

Consequently, the norm $|\boldsymbol{\omega}|$ of $\boldsymbol{\omega}$ is related to θ as

$$|\boldsymbol{\omega}| = 2\tan\frac{\theta}{2}. \tag{8}$$

In order to avoid singularity, Rankin and Brogan use a different definition of the transformation matrix, that is given by

$$\mathbf{T} = \mathbf{I} + \boldsymbol{\Omega}\sqrt{1-\frac{1}{4}\left|\boldsymbol{\omega}\right|^2} + \frac{1}{2}\boldsymbol{\Omega}^2, \tag{9}$$

and, in this case, the relationship between $|\boldsymbol{\omega}|$ and θ is

$$|\boldsymbol{\omega}| = 2\sin\frac{\theta}{2}. \tag{10}$$

This equation shows the limit of Rankin and Brogan's corotational procedure. Using the elements of $\boldsymbol{\omega}$ as rotational degrees of freedom, the magnitude $|\boldsymbol{\omega}|$ can approximate the angle θ only for rotations up to 30°.

The algorithm proposed by Rankin and Brogan can be summarized as follows. Given the configuration of a finite element at step k, local translational displacements are calculated using Eq. 1. A relative rotation matrix $\mathbf{T}_k$ is calculated using Eq. 2. The rotational freedoms are taken from the pseudovector $\boldsymbol{\omega}_k$ that is associated with the skew-symmetric matrix

$$\boldsymbol{\Omega}_k = \frac{2(\mathbf{T}_k - \mathbf{T}_k^{\mathrm{T}})}{1 + \mathrm{tr}(\mathbf{T}_k)}. \tag{11}$$

Internal forces calculated within the element are then transformed to the global coordinate system. Displacement increments required to reduce the out-of-balance forces are then obtained by solution of the system of equations. While the translational increments are directly added to the total displacement vector, the rotational increments $\Delta\boldsymbol{\omega}$ are used to update the nodal *pseudovector* $\boldsymbol{\omega}_k$ using the following equation:

$$\boldsymbol{\omega}_{k+1} = \pm \frac{\Delta\boldsymbol{\omega}\sqrt{1 - \frac{1}{4}\left|\boldsymbol{\omega}_k\right|^2} + \boldsymbol{\omega}_k - \frac{1}{2}\boldsymbol{\omega}_k \times \Delta\boldsymbol{\omega}}{\sqrt{1 + \frac{1}{4}\left|\boldsymbol{\omega}_k\right|^2}}, \tag{12}$$

where the sign must be the same as the sign of the quantity

$$\sqrt{1 - \frac{1}{4}\left|\boldsymbol{\omega}_k\right|^2} - \frac{1}{4}\boldsymbol{\omega}_k \cdot \Delta\boldsymbol{\omega}. \tag{13}$$

Note that the *pseudovector* $\boldsymbol{\omega}_k$ defines the relative rotation of the surface coordinate system with respect to the element coordinate system. The global rotations are not updated using the pseudovector approach, but the rotation increments are simply added to the rotations at the previous step. A complete discussion on the corotational procedure can be found in [1] and [15].

As previously pointed out, the corotational procedure has been widely used and implemented in general purpose finite element codes (e.g. STAGS, ANSYS). It was demonstrated that this procedure is efficient and accurate in large rotation/small strain problems. When

the strain within the element becomes larger, this procedure does not perform well. It was shown in this section that the element local rotations (the rotations of the surface coordinate system relative to the element coordinate system) must remain less than 30°, otherwise the approximation of the element rotational degrees of freedom with the *pseudovector* components can not be considered accurate. Furthermore, in order to define correctly the element equations in the current element configuration, the element displacement increments must be small. While these restrictions may not be serious in structural applications, especially when very small load/time steps are used, serious problems can be encountered when simulating simple dynamic multibody problems as demonstrated by the results presented in later sections.

3 NON INCREMENTAL ABSOLUTE NODAL COORDINATE FORMULATION

In this section, the non-incremental finite element absolute nodal coordinate formulation [19] is briefly reviewed. In this formulation, the nodal coordinates of the element are defined in a fixed inertial coordinate system, and consequently no transformation is required for the element coordinates. The element nodal coordinates represent global nodal displacements and slopes. Thus, in the absolute nodal coordinate formulation, no infinitesimal or finite rotations are used as nodal coordinates and no assumption on the magnitude of the element rotations is made. The results presented in this paper will demonstrate that these properties of the absolute nodal coordinate formulation make this formulation efficient and accurate in the large displacement analysis of flexible multibody systems.

In this investigation, two dimensional beam elements are considered. The global position vector $\mathbf{r}$ of an arbitrary point P on the element is defined in terms of the nodal coordinates

and the element shape function, as shown in Fig. 2, as

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \mathbf{Se}, \tag{14}$$

where $\mathbf{S}$ is the global shape function which has a complete set of rigid body modes, and $\mathbf{e}$ is the vector of element nodal coordinates:

$$\mathbf{e} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{bmatrix}^{\mathrm{T}}. \tag{15}$$

This vector of absolute nodal coordinates includes the global displacements

$$e_1 = r_1\big|_{x=0}, \quad e_2 = r_2\big|_{x=0}, \quad e_5 = r_1\big|_{x=l}, \quad e_6 = r_2\big|_{x=l}, \tag{16}$$

and the global slopes of the element nodes, that are defined as

$$e_3 = \left.\frac{\partial r_1}{\partial x}\right|_{x=0}, \quad e_4 = \left.\frac{\partial r_2}{\partial x}\right|_{x=0}, \quad e_7 = \left.\frac{\partial r_1}{\partial x}\right|_{x=l}, \quad e_8 = \left.\frac{\partial r_2}{\partial x}\right|_{x=l}. \tag{17}$$

Here x is the coordinate of an arbitrary point on the element in the undeformed configuration, and l is the original length of the beam element. Since absolute coordinates are used, a cubic polynomial is employed to describe both components of the displacements. Therefore, the global shape function $\mathbf{S}$ can be written as

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 & s_2 l & 0 & s_3 & 0 & s_4 l & 0 \\ 0 & s_1 & 0 & s_2 l & 0 & s_3 & 0 & s_4 l \end{bmatrix}, \tag{18}$$

where the functions $s_i = s_i(\xi)$ are defined as

$$s_1 = 1 - 3\xi^2 + 2\xi^3, \quad s_2 = \xi - 2\xi^2 + \xi^3, \quad s_3 = 3\xi^2 - 2\xi^3, \quad s_4 = \xi^3 - \xi^2, \tag{19}$$

and $\xi = x/l$. It can be shown that the preceding shape function contains a complete set of rigid body modes that can describe arbitrary rigid body translational and rotational displacements, provided that global slope coordinates are used instead of infinitesimal rotations. Using the absolute coordinates and slopes, it can also be shown that the beam element defined by the shape function of Eq. 18 is an isoparametric element.

4 PERFORMANCE IN STATIC PROBLEMS

In this section, the performance of the two incremental and non-incremental finite element procedures discussed in the preceding two sections in the static analysis of large deflection problems of planar beams is investigated using two numerical examples. These two examples of large static deformation are solved using the general purpose finite element code ANSYS that utilizes the corotational procedure presented by Rankin and Brogan [15] for the large deformation problems [1], and they are also solved using the non-incremental absolute nodal coordinate formulation [7]. The results of ANSYS and the absolute nodal coordinate formulation are obtained using a linear strain-displacement relationship. The first example is a cantilever beam loaded with a free end moment that bends into a full circle, while the second problem is the elastica problem. Both examples, which have a known analytical solution, employ the same beam model. The beam in this model is assumed to have length of 1 m, cross sectional area of 1.257E-03 m^2, second moment of area of 1.257E-07 m^4, and modulus of elasticity of 2.0E+09 Pa.

Bending of a Beam into a Full Circle This example is shown in Fig. 3. The beam is divided into 10 elements and the results obtained using ANSYS are almost identical to the exact solution. The total CPU time required to obtain the solution shown in Fig. 3 using ANSYS on HP-Convex SPP1200/XA-16 was found to be 7.5 sec. The same results were also obtained by Rankin and Brogan, who emphasized that the corotational procedure gives good results in this kind of analysis because the large deflections of the beam are converted into much smaller deformational increments at each load step. They also showed that the results obtained using a conventional incremental approach were not accurate and that no solution could be obtained when the free end rotation reached about 90° [15]. In the example of Fig. 3, the ratio between the nodal rotation increments and the load increments is constant. Furthermore, the curvature of the beam remains constant along its length as shown in Fig. 3. Because of these characteristics, the load step increments are easily generated such that

the rotational increments lie within the range of allowed rotations.

The absolute nodal coordinate formulation was also used to solve the same problem. Figure 4 shows the results of the global rotation of the beam free end in the full circle example. The results obtained using the absolute nodal coordinate formulation are compared with the exact solution. In Fig. 5, the solution configurations of the beam loaded by the free end moment are presented. These results were obtained by dividing the beam into 10 finite elements. The total CPU time for obtaining this solution on a PC Pentium 90MHz was found to be 7.2 sec.

Figure 4 shows that the absolute nodal coordinate formulation gives slightly different results from the exact solution (6% error) when $\theta_{free\ end} = 360°$. Nonetheless, the solution obtained using the absolute nodal coordinate formulation is quite accurate and the CPU time demonstrates that the method is computationally efficient. The difference from the exact solution is due to the fact that a linear elastic model was used for the formulation of the elastic forces. It can be demonstrated that a better solution is obtained using a higher number of elements.

Elastica Problem The second example of static analysis of beam large deflection problem is the *elastica* problem in which a cantilever beam is subject to a compressive load at the free end [7]. The analytical solution of this problem can be found in Timoshenko [21]. Figure 6 shows the deformed configurations of the beam predicted using ANSYS under different loads over the critical limit for the cases of 10 and 20 finite elements discretization. In Table 1, the results of the global rotations of the free end node obtained using ANSYS are presented and compared to the exact solutions. The CPU time for obtaining the solution of Fig. 6 on an HP-Convex SPP1200/XA-16 was 60 sec and 64 sec for 10 and 20 finite element cases, respectively. The solution obtained using ANSYS is very close to the exact solution in the range $40°-120°$of rotation of the free end. Rankin and Brogan [15] affirmed that no solution to the elastica problem could be obtained using the conventional incremental

approach. When the load approaches the critical value (for free end rotations $< 40°$), the results from the ANSYS solution are different from the exact solution. This is due to the fact that in the vicinity of the critical value, a very small change of load causes a very large change in solution configurations. In fact, when the tolerance parameter for the equilibrium iterations is decreased for a better iterative refinement, a solution much closer to the exact one can be obtained in the range of solutions with free end rotations $< 40°$. However, it was impossible to obtain a solution using ANSYS for loads that give free end rotations $> 140°$ using both 10 and 20 elements. The solution for a load which causes an exact solution of $140°$ of free end rotation is also inaccurate. Close analysis of this problem showed that for loads that give free end rotations $\geq 140°$ the curvature of the beam in the neighborhood of the fixed end becomes quite large, while the rest of the beam remains almost straight. When the curvature becomes large, the relative rotation is greater than $30°$. Furthermore, the automatic load stepping is governed by the average element rotational increments that in this example remain small even though the curvature in the neighborhood of the fixed end is relatively large. Consequently, it becomes very difficult for the ANSYS code to adjust the load step according to the increase of the curvature in a very limited area of the beam. This example shows that using the rotations as nodal degrees of freedom leads eventually to accuracy and convergence problems.

The same elastica problem was solved using the absolute nodal coordinate formulation. Table 2 shows the results of the global rotation of the beam free end in the elastica problem using the absolute nodal coordinate formulation with 10 and 20 finite elements [7]. The solution configurations under different overcritical loads are shown in Fig. 7. These results show that the solution obtained using the absolute nodal coordinate formulation is very close to the exact solution. For loads close to the critical value (solutions with free end rotations $\leq 40°$), the case of 10-element discretization gives better results than the 20-element case, because the solution is obtained for a smaller number of variables while the

elastic deformation of the beam remains relatively small. Unlike the results obtained using ANSYS, accurate solutions are also obtained for free end rotations $\geq 140°$. In these cases, the 20-element model performs better than the 10-element model, since the more refined discretization of the beam can better describe the large deformations. The total CPU time used to obtain the solution on a Pentium 90 MHz was 8 sec and 14 sec for the 10 and 20-element models, respectively. These results demonstrate that the non-incremental absolute nodal coordinate formulation leads to an accurate and efficient solution as compared to the incremental corotational procedure.

5 DYNAMIC PROBLEMS

The results presented in the preceding section demonstrate that the non-incremental absolute nodal coordinate formulation performs well in static problems despite the fact that such a formulation leads to a complex expression for the elastic forces. In fact, it is surprising to note that the absolute nodal coordinate formulation is more efficient as compared to the incremental methods in static applications, and this formulation leads to accurate results in the large deformation problems by using a linear strain-displacement relationship. Since the absolute nodal coordinate formulation leads to a constant mass matrix, it is expected that this formulation will perform even better in dynamics problems.

Incremental Finite Element Approach The incremental finite element approach has been widely used for the dynamic analysis of flexible systems that undergo large rotations and deformations. In the incremental finite element formulation, nodal rotations are used as degrees of freedom and the nodal coordinates are treated as vectors [2, 10, 15]. The internal forces of the flexible bodies are first defined in the element coordinate systems and then transformed to the global system. The dynamic equations are then solved for the deformation increments. In Section 2, a corotational procedure was presented which allows

the use of the conventional finite element formulations in large rotation problems. In the dynamics of flexible bodies that undergo large rotations, it is important to obtain accurate modeling of the inertia of the bodies. However, it was recently demonstrated [17] that the use of the incremental approach where rotations are used as nodal coordinates does not lead to the exact modeling of the rigid body dynamics of simple structures.

When the incremental formulations are used with consistent mass techniques, the global mass matrix of the element is not constant. As a consequence, the expression of the inertia forces does not take a simple form and these forces have to be updated at every time step. In the ANSYS code, the conventional shape function of the beam element is used. In this shape function, the axial displacement is approximated using a linear polynomial, while the transverse displacement is approximated using a cubic polynomial. This is the displacement field which is used to generate the ANSYS results presented in the following section.

Absolute Nodal Coordinate Formulation In Section 3, the generalized nodal coordinates and the displacement field of the absolute nodal coordinate formulation were presented. In this formulation, the global position vector of an arbitrary point on the element is defined in terms of a set of global nodal coordinates and a global shape function. It is assumed that this global shape function has a complete set of rigid body modes. By differentiating Eq. 14 with respect to time, we obtain the global velocity vector that can be used to define the kinetic energy of the element as

$$T = \frac{1}{2}\int_V \rho \dot{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}dV = \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\left(\int_V \rho \mathbf{S}^{\mathrm{T}}\mathbf{S}dV\right)\dot{\mathbf{e}}, \tag{20}$$

where ρ and V are, respectively, the mass density and volume of the element. We can define the mass matrix of the element as

$$\mathbf{M} = \int_V \rho \mathbf{S}^{\mathrm{T}}\mathbf{S}dV, \tag{21}$$

where $\mathbf{M}$ is a symmetric and constant mass matrix and it is the same matrix used in linear structural dynamics. Using the global shape function defined in Eq. 18, the mass matrix of

the element can be written as

$$\mathbf{M} = m \begin{bmatrix} \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & \frac{-13l}{420} & 0 \\ & \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & \frac{-13l}{420} \\ & & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & \frac{-l^2}{140} & 0 \\ & & & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & \frac{-l^2}{140} \\ & & & & \frac{13}{35} & 0 & \frac{-11l}{210} & 0 \\ & & & & & \frac{13}{35} & 0 & \frac{-11l}{210} \\ & & sym. & & & & \frac{l^2}{105} & 0 \\ & & & & & & & \frac{l^2}{105} \end{bmatrix} \tag{22}$$

where m is the mass of the beam element and l is its length. Note that a consistent mass approach has been used in defining the mass matrix. It can be demonstrated that this mass matrix leads to exact modeling of the rigid body inertia, while a lumped mass approach would lead to a wrong modeling of the inertia. Using the expressions of the elastic and inertia forces previously obtained, in the absolute nodal coordinate formulation the equations of motion of the finite element take the following simple form:

$$\mathbf{M}\ddot{\mathbf{e}} + \mathbf{Q}_k = \mathbf{Q}_a, \tag{23}$$

where $\mathbf{Q}_k$ is the vector of the elastic forces, and $\mathbf{Q}_a$ is the vector of applied nodal forces. While the mass matrix is a constant matrix, the vector of elastic forces is highly nonlinear function of the absolute nodal coordinates. The preceding equation can be written as

$$\mathbf{M}\ddot{\mathbf{e}} = \mathbf{Q}, \tag{24}$$

where the vector $\mathbf{Q} = \mathbf{Q}_a - \mathbf{Q}_k$. Since the mass matrix is constant, efficient and accurate numerical procedures can be used to solve the preceding system of equations for the vector of the generalized accelerations $\ddot{\mathbf{e}}$. For instance, a Cholesky decomposition of the symmetric positive definite mass matrix can be made once at the beginning of the integration and used throughout the entire numerical solution.

6 PERFORMANCE IN DYNAMIC APPLICATIONS

Twoproblems have been investigated in order to compare between the performances of the corotational procedure and the absolute nodal coordinate formulation in dynamic applications. As expected, problems are encountered when the corotational procedure is used to solve some simple multibody applications. Since the ANSYS code can not be used to model complex multibody problems, in this section only simple multibody applications are considered. These applications are: (1) free falling of a pendulum, and (2) non-smooth motion of a four bar mechanism.

Pendulum Problem The first dynamic problem considered in this section is the free falling of a very flexible two dimensional beam under the effect of gravity. The beam is connected to the ground by a pin joint at one end, as shown in Fig. 8. The beam has a length of 1.2 m, a circular cross section with an area of 0.0018 m^2, a second moment of area of 1.215E-08 m^4, and a modulus of elasticity of 0.700E+06 Pa. In the original configuration, the beam is horizontal and has zero initial velocity.

Two cases are considered in the analysis of the falling pendulum. In the first case, the beam is assumed to fall under the effect of gravity, while in the second case the beam is accelerated by increasing the gravity constant to 50 m/s^2. The results of the two models of the pendulum are obtained using the absolute nodal coordinate formulation and the corotational procedure proposed by Rankin and Brogan [15]. Three models were considered to simulate the motion of the free falling pendulum. These models employ 12, 40 and 100 finite elements. The configurations of the free falling pendulum at different time steps predicted using the absolute nodal coordinate formulation and the 12-element model are shown in Fig. 9. Figure 10 shows the transverse deflection of the midpoint of the pendulum using the three different models. From the results presented in this figure it is clear that the 12-element solution leads to accurate results. The solutions obtained using the 40 and 100-element models are identical.

In the case of the large value of the acceleration constant, the deformation of the beam becomes much larger. The configurations of the pendulum at different time steps predicted using the absolute nodal coordinate formulation and the 40-element model are shown in Fig. 11. In this case, the 12 element solution does not lead to very accurate results, due to the large deformation. However, the results obtained using 40 elements are accurate, as shown by Fig. 12 where the transverse deflection of the midpoint of the beam is plotted versus time.

Figure 13 shows the transverse deflection of the mid point of the pendulum obtained using ANSYS when the gravity constant is equal to 9.81 m/s^2. The 12-element model does not converge. Furthermore, the 40-element solution diverges after 0.7 sec despite the simplicity of the model. Only when a large number of elements is used, convergence is achieved using the corotational formulation. It is important to point out that changing the number of steps and the number of convergence iterations does not result in an improvement of the results. This convergence problem is attributed to the use of local rotations as nodal coordinates in the corotational formulation. In this problem, the relative rotation between the surface coordinate system and the element coordinate system becomes larger than 30° when a small number of elements is used. This leads to problems when the corotational formulation is used, as explained in Section 2.

When the gravity constant is increased to 50 m/s^2, the corotational procedure fails in the simulation of the motion of the simple pendulum. In this case, a high value of the acceleration and a relatively high mass produce large inertia forces, and this results in large deformations and large angular velocities. The high inertia forces and angular velocities, which are characteristics of multibody applications, pose serious problems when the corotational formulations are used. As demonstrated by the results presented in Fig. 14, 100 elements are not enough to achieve convergence, and the solution diverges after 0.3 sec.

Four Bar Mechanism The simulation results of the simple pendulum previously presented in this section clearly demonstrate some of the serious problems that can be encountered in the simulations of very simple multibody systems as the deformation and speed increase. In this section, another multibody example, the four bar mechanism shown in Fig. 15, is considered. The dimensions and material properties of the links of the four bar mechanism are shown in Table 3. All components of the mechanism are made of steel and have a circular cross section with diameter equal to 0.4 m. This system is designed to obtain high values of the angular velocities of the connecting rod and the follower as compared to the angular velocity of the crankshaft. In this system, complete rotations of the crankshaft are possible, as the Grashoff's law gives:

$$s + l = 1.2 \leq 1.21 = p + q, \tag{25}$$

where s and l are the lengths of the shortest and longest links, and p and q are the lengths of the other two links. However, the difference between the two sides of Eq. 25 is very small, and this makes the motion non-smooth. In the case of rigid body motion, the angular velocities of the connecting rod and the follower are presented in Fig. 16 as functions of the angle of rotation of the crankshaft assuming a unit value for the angular velocity of the crankshaft. It is clear from the results presented in this figure that when the rotation of the crankshaft is close to 0, 2π, 4π, ... the angular velocities of the connecting rod and the follower change dramatically in a very short time. The system is assumed to be driven by a moment, shown in Fig. 17 as a function of time, applied to the crankshaft, and the effect of the gravity force is taken into consideration.

Figure 18 shows the transverse deflection of the midpoint of the connecting rod predicted using the absolute nodal coordinate formulation. The transverse deflection is determined as the distance of the midpoint from a straight line that connects the two ends of the connecting rod. It is clear from the results presented in Fig. 18 that up to approximately 0.75 sec the motion is very smooth and the deformation of the connecting rod remains small. After 0.75

sec, the crankshaft completes a full revolution and there is a jump in the angular velocity leading to a large deformation.

Several simulations have been performed using the corotational formulation implemented in ANSYS, using different numbers of steps in the integration routine. In the first simulation, 20,000 time steps were chosen while maintaining the option of automatic stepping active. This simulation configuration leads to the solution shown in Fig. 19, where the global vertical position of point A on the crankshaft is presented and compared to the solution obtained using the absolute nodal coordinate formulation. Before 0.75 sec there is no difference between the two solutions, but after that the two curves diverge. It is clear that for the corotational procedure to converge, a smaller integration step is required. In order to achieve this, the automatic stepping option is removed in a second simulation. This change improves the results significantly, as demonstrated by the results shown in Fig. 20. However, there are still differences when the deflections are considered instead of global positions of nodes, as demonstrated by the results shown in Fig. 21. Furthermore, increasing the number of time steps to 40,000 does not lead to a better improvement of the results, as shown by the results of Fig 22.

In this four bar mechanism problem, the total deformation of the bodies remains small, but the angular velocity experiences jumps each time the crankshaft completes a full cycle. Hence, in the vicinity of that configuration, the displacement increments are large within a single time step, and the results given by the corotational procedure are not accurate. Theoretically, convergence can be achieved as the time step approaches zero. This, however, may lead to excessive error accumulation in practice. In fact, Fig. 22 shows that the connecting rod has the same pattern of vibration as previously predicted by the absolute nodal coordinate formulation. Nonetheless, in the case of the corotational procedure implemented in ANSYS, after about 1 sec, a phase shift develops, and this shift cannot be corrected with a further increase in the number of time steps.

7 SUMMARY AND CONCLUSIONS

Many multibody applications are characterized by motion discontinuities, high inertia forces and high and discontinuous angular velocities. In this investigation, two finite element procedures, the corotational technique and the absolute nodal coordinate formulation, which can be used for the solution of large deformation problems, are presented and their computational performances are demonstrated using several numerical examples. In this investigation, the limitations of the corotational formulation, that has been implemented in several general purpose finite element codes, are demonstrated when flexible multibody applications are considered. It is shown that the incremental procedure can be computationally expensive in large deflection problems as compared to the non-incremental absolute nodal coordinate formulation in which the nodal coordinates are defined in a fixed inertial frame. The absolute nodal coordinate formulation leads to a constant mass matrix which is the same as the mass matrix used in linear structural analysis. Therefore, the inertia forces are linear functions in the accelerations and the dynamic equations of motion do not include any quadratic velocity terms. The elastic forces, on the other hand, are highly nonlinear function of the nodal coordinates even in the case of linear elastic models.

In the case of static analysis of beam large deflection problems, it is demonstrated that the absolute nodal coordinate formulation leads to accurate results. On the other hand, it is shown that the corotational procedure can be computationally expensive and can lead to a lock in the solution because of the presence of the rotations in the set of nodal coordinates. The performance of the absolute nodal coordinate formulation in dynamic problems has also been evaluated using a flexible pendulum and a flexible four bar mechanism. Due to the limitations on the amplitudes of the rotations in the corotational procedure, such a formulation can fail in the simulation of simple multibody systems, as demonstrated by the results presented in this study. In applications characterized by high inertia forces and motion and velocity discontinuities, serious problems can be encountered in the simulation

of flexible multibody systems.

It was demonstrated that the results obtained using the absolute nodal coordinate formulation agree well with the results obtained using the *floating frame of reference formulation* in the case of small deformation problems [7, 19]. The absolute nodal coordinate formulation, however, can be used as the basis for developing a new generation of flexible multibody codes that can be used in the small and large deformation analysis of flexible multibody systems, as demonstrated in this investigation.

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Table 1. Elastica problem: global rotations of the free end node. Exact and ANSYS solutions

P / P_{cr} (*)	1.015	1.063	1.152	1.293	1.518	1.884	2.541	4.029	9.116
$\theta_{free-end}$ exact	20°	40°	60°	80°	100°	120°	140°	160°	180°
$\theta_{free-end}$ 10 elements	33.25°	41.51°	59.35°	79.53°	99.82°	119.98°	130.77°	-	-
$\theta_{free-end}$ 20 elements	33.85°	42.15°	59.80°	79.79°	99.93°	119.98°	131.84°	-	-

(*) $P_{cr} = \dfrac{\pi^2 EI}{4l^2}$

Table 2. Elastica problem: global rotations of the free end node. Exact and absolute nodal coordinate formulation solutions

P / P_{cr}	1.015	1.063	1.152	1.293	1.518	1.884	2.541	4.029	9.116
$\theta_{free-end}$ exact	20°	40°	60°	80°	100°	120°	140°	160°	180°
$\theta_{free-end}$ 10 elements	21.41°	38.56°	58.80°	78.55°	98.53°	118.48°	138.56°	158.88°	175.57°
$\theta_{free-end}$ 20 elements	22.53°	39.74°	60.11°	80.04°	100.17°	120.20°	140.22°	160.22°	176.11°

Table 3. Parameters used in the simulation of the four-bar mechanism

Body	m [kg]	A [m^2]	I [m^4]	l [m]	E [Pa]
Crankshaft	4.9323	1.257E-03	1.257E-07	0.5	2.1E+11
Coupler	6.9052	1.257E-03	1.257E-07	0.7	2.1E+11
Follower	5.5242	1.257E-03	1.257E-08	0.56	2.1E+11

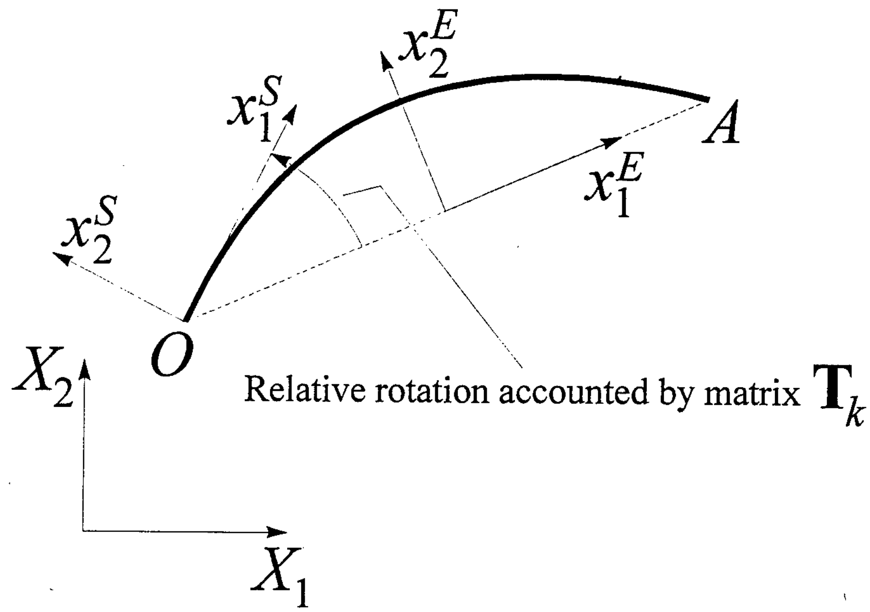
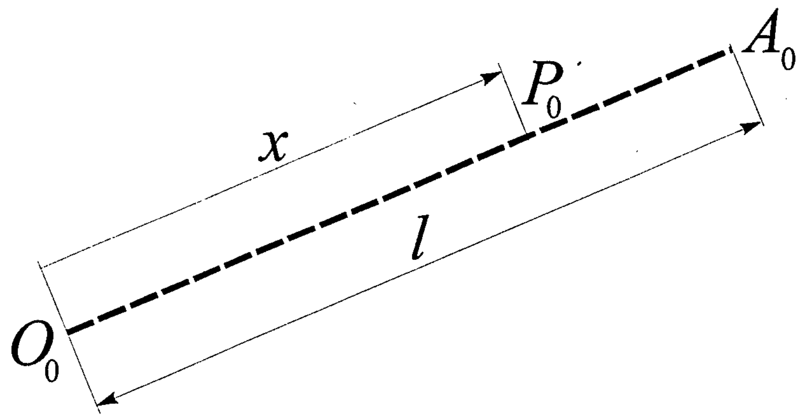
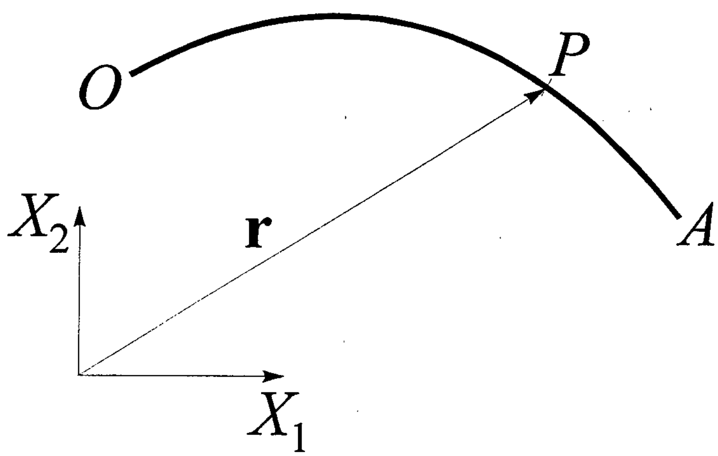


Fig. 1. Corotational procedure



a) Undeformed configuration



a) Deformed configuration

Fig. 2. Absolute nodal coordinate formulation

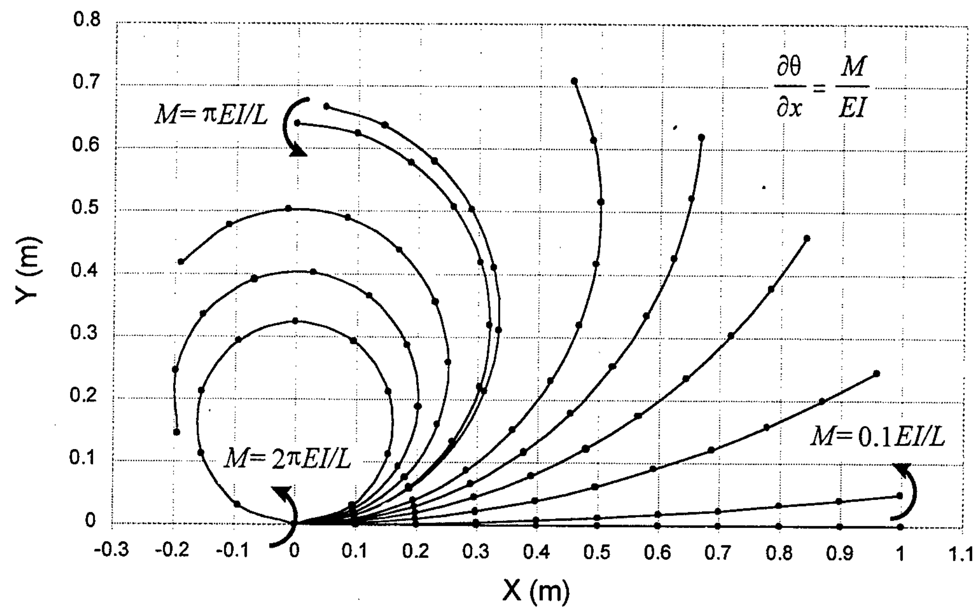


Fig. 3. Cantilever beam bent into a full circle by an end moment. ANSYS solution

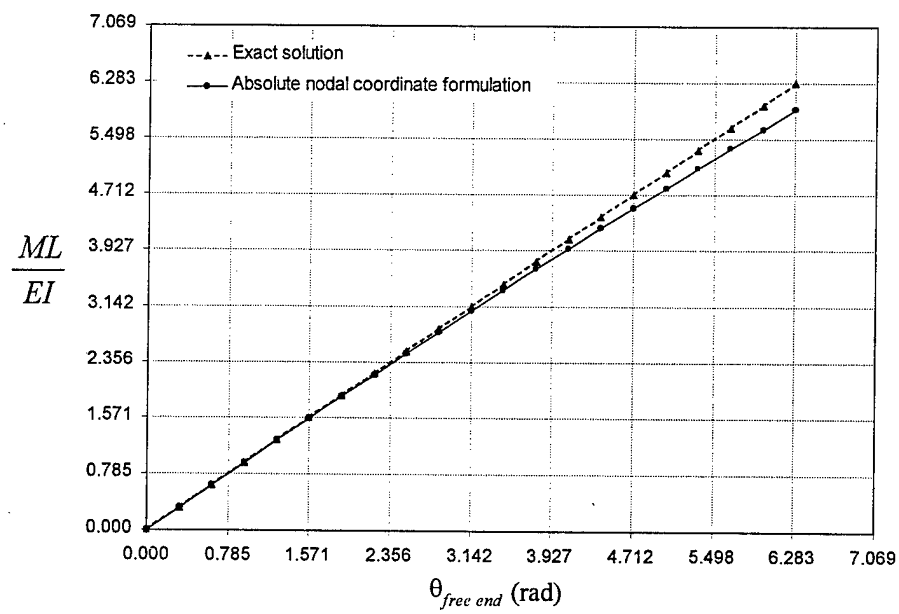


Fig. 4. Rotations of the free end node of a cantilever beam subject to end moments

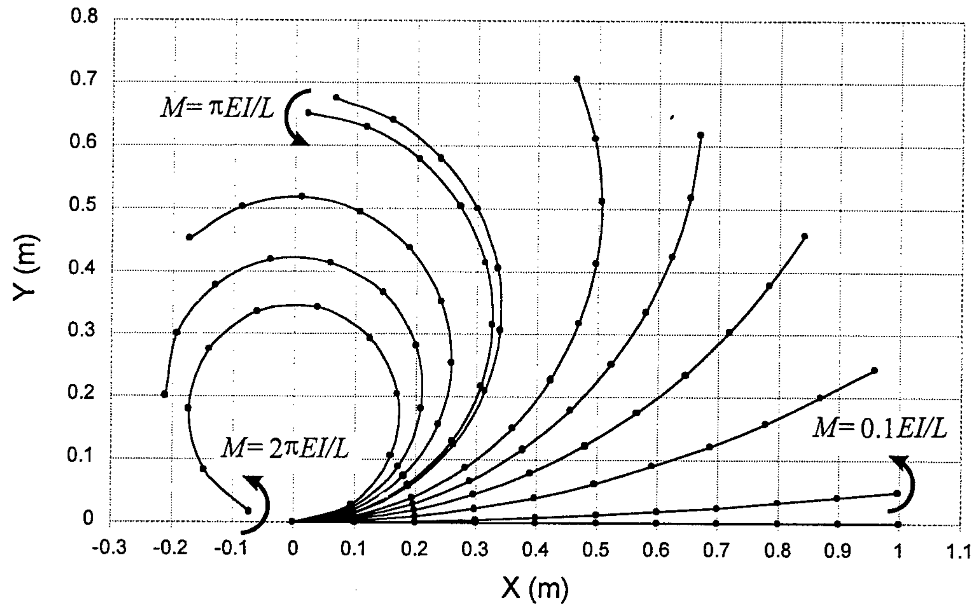


Fig. 5. Cantilever beam bent into a full circle by an end moment. Absolute nodal coordinate formulation solution

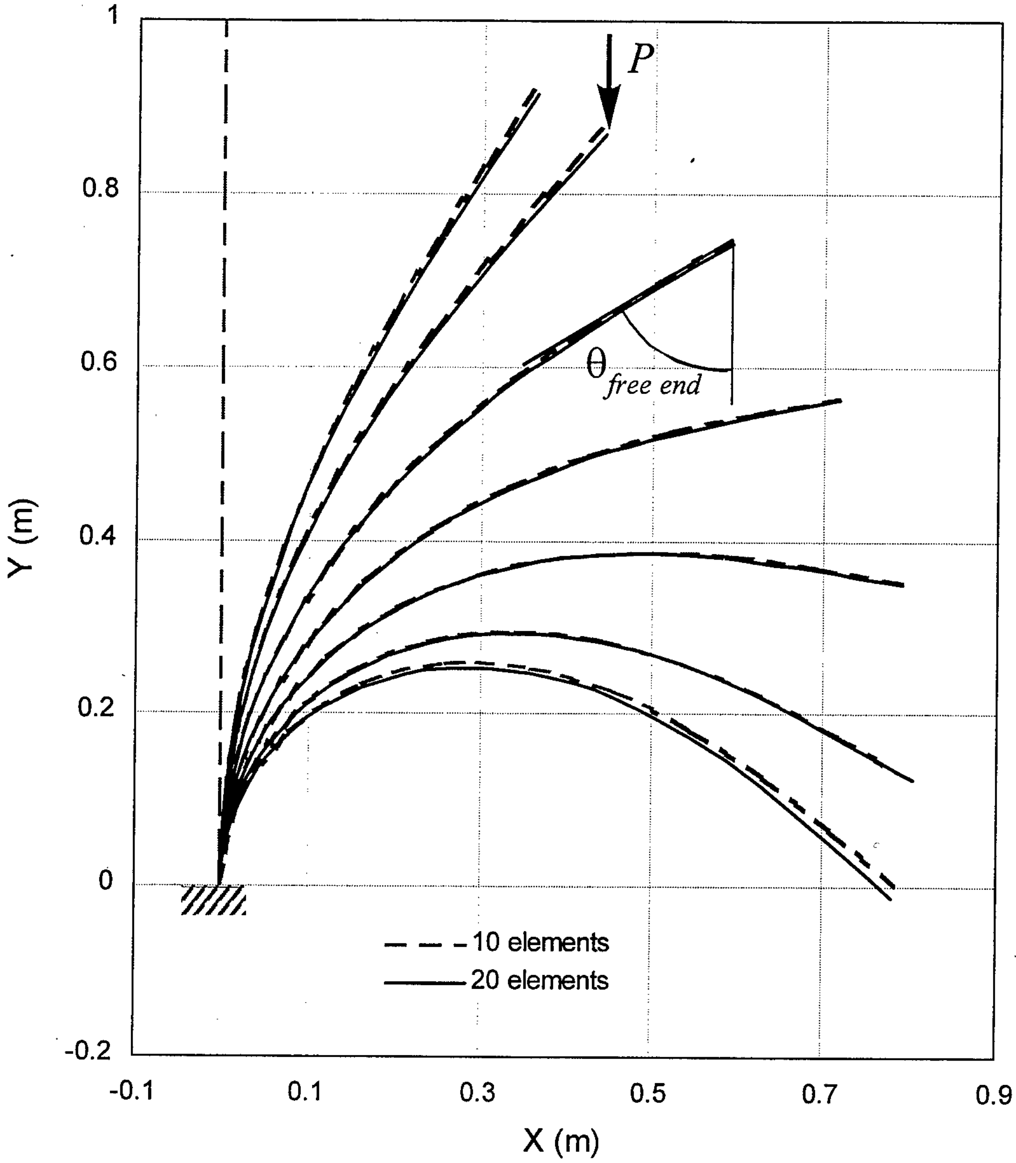


Fig. 6. Deformed shapes of the cantilever beam subject to overcritical loads. Solutions obtained using ANSYS

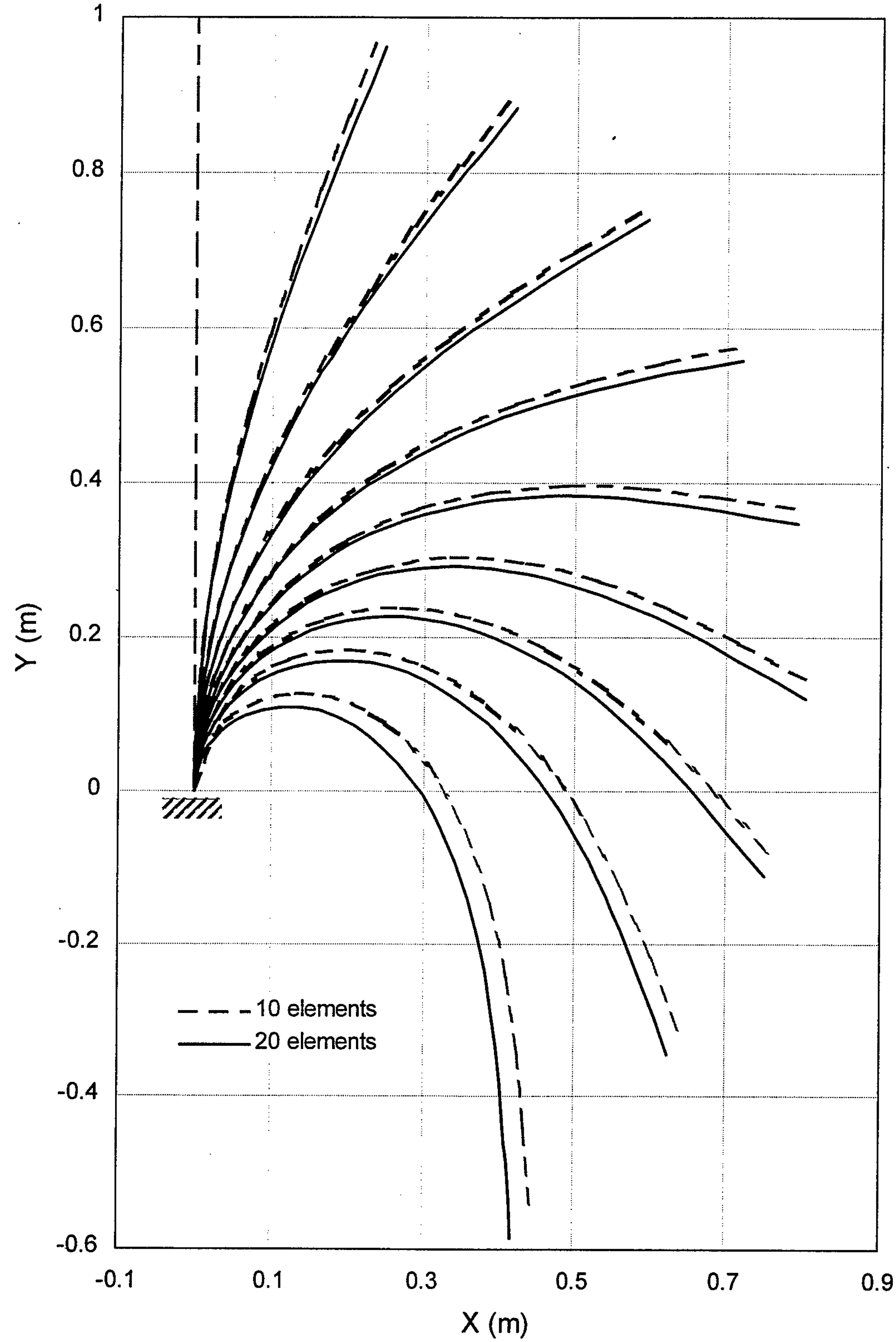


Fig. 7. Deformed shapes of the cantilever beam subject to overcritical loads. Solutions obtained using the absolute nodal coordinate formulation

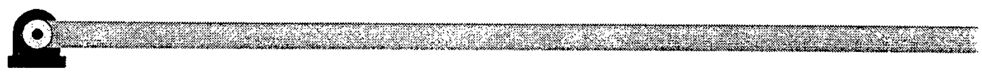


Fig. 8. Free falling pendulum

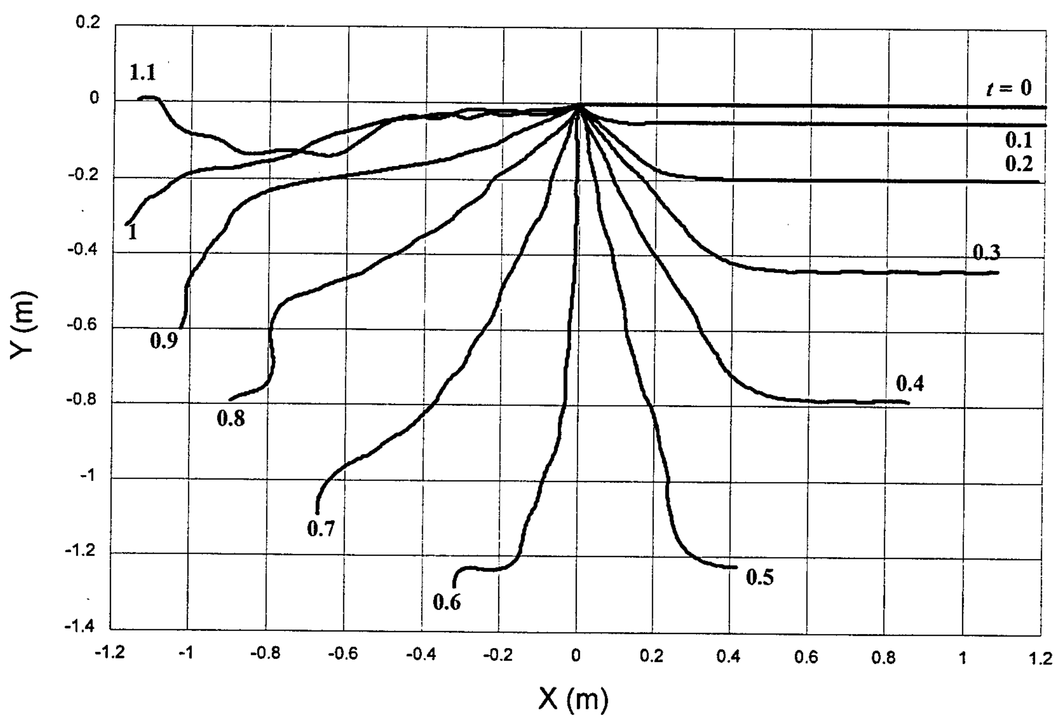


Fig. 9. Configurations of the free falling pendulum at different times for the case a=9.81 m/s²
(values of time given in sec)

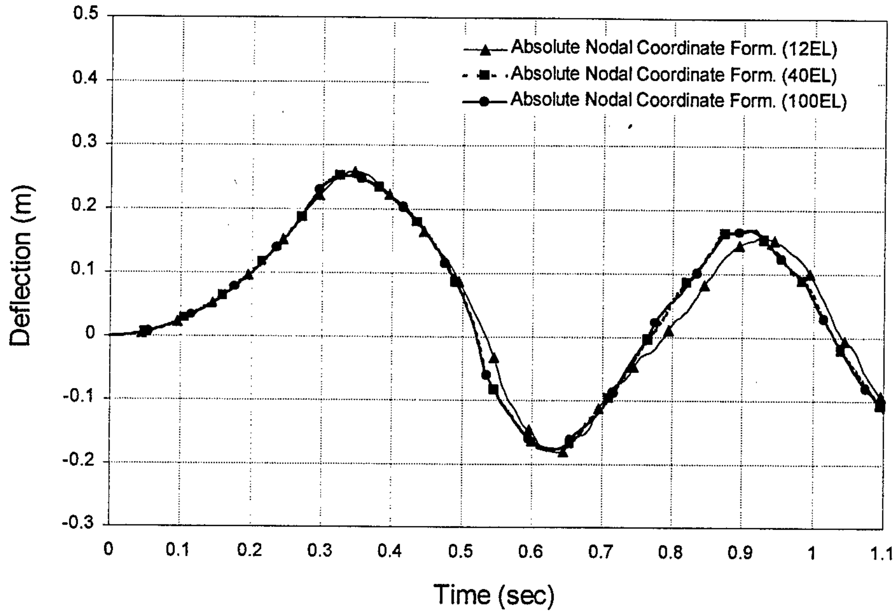


Fig. 10. Transverse deflection of the midpoint of the pendulum for different models. (a=9.81 m/s^2)

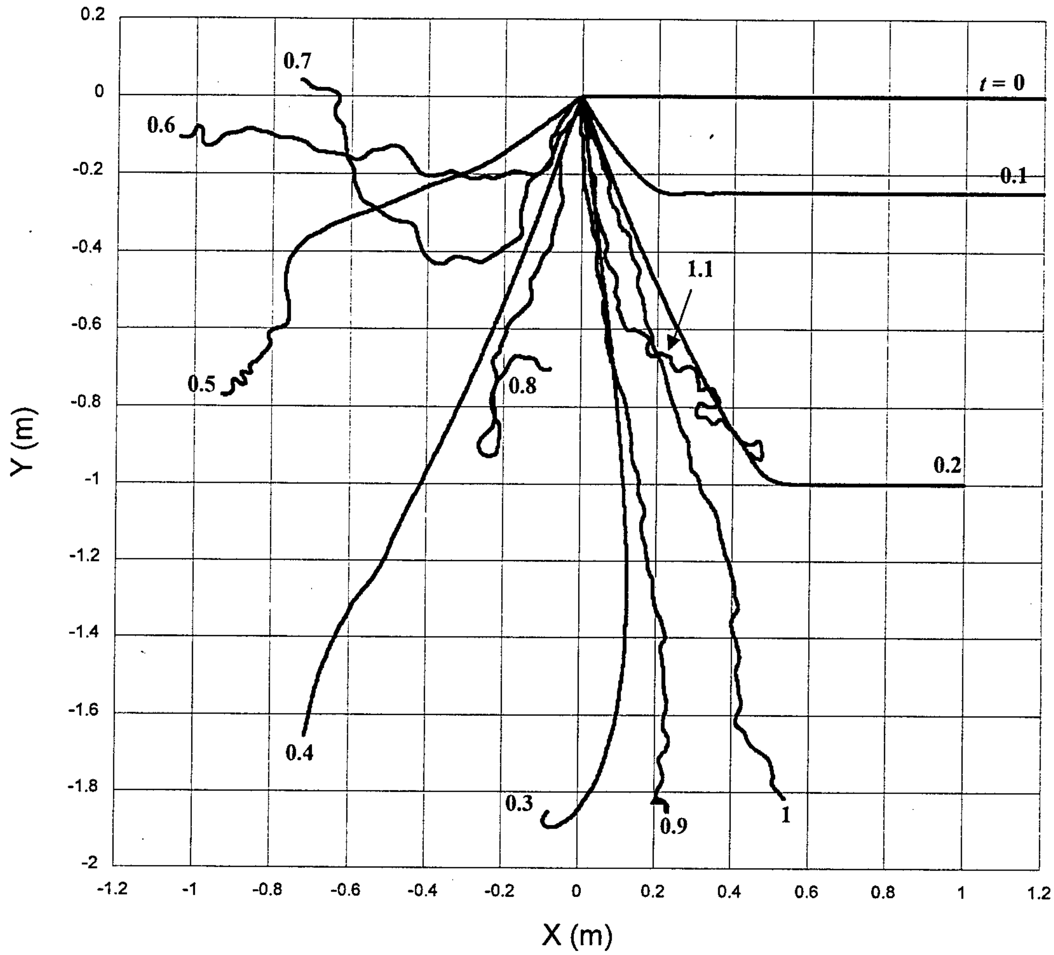


Fig. 11. Configurations of the free falling pendulum at different times for the case a=50 m/s^2 (values of time given in sec)

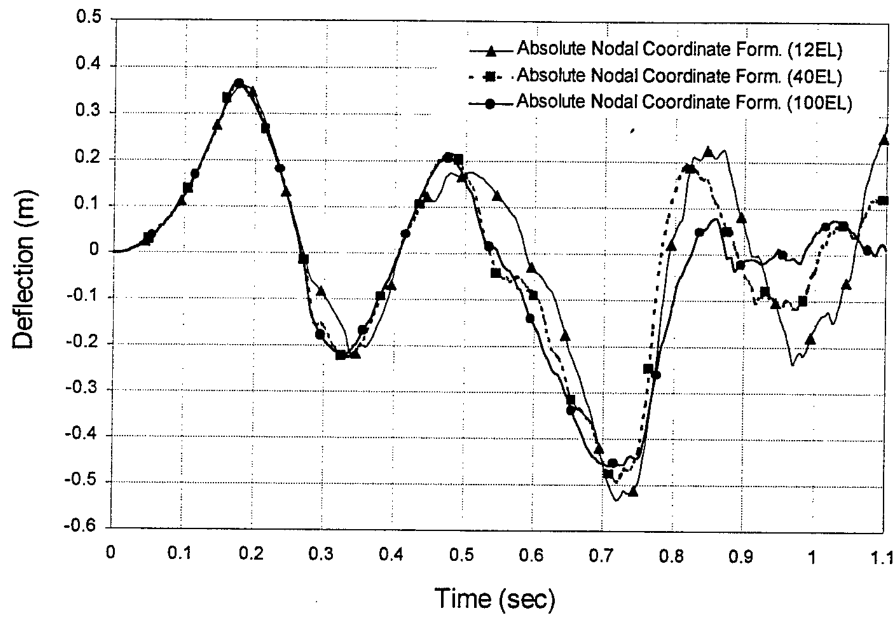


Fig. 12. Transverse deflection of the midpoint of the pendulum for different models. (a=50 m/s²)

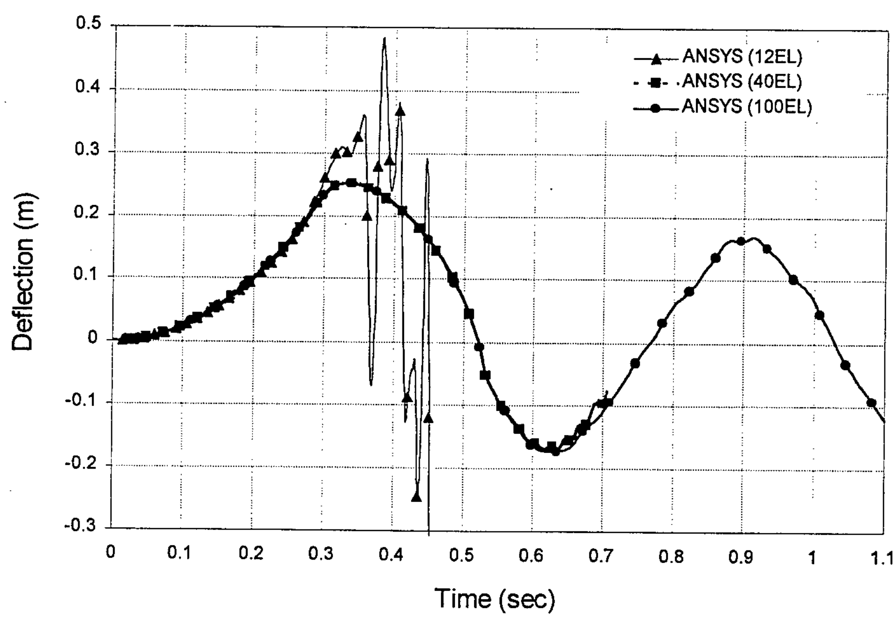


Fig. 13. Transverse deflection of the midpoint of the pendulum for different models. (a=9.81 m/s²)

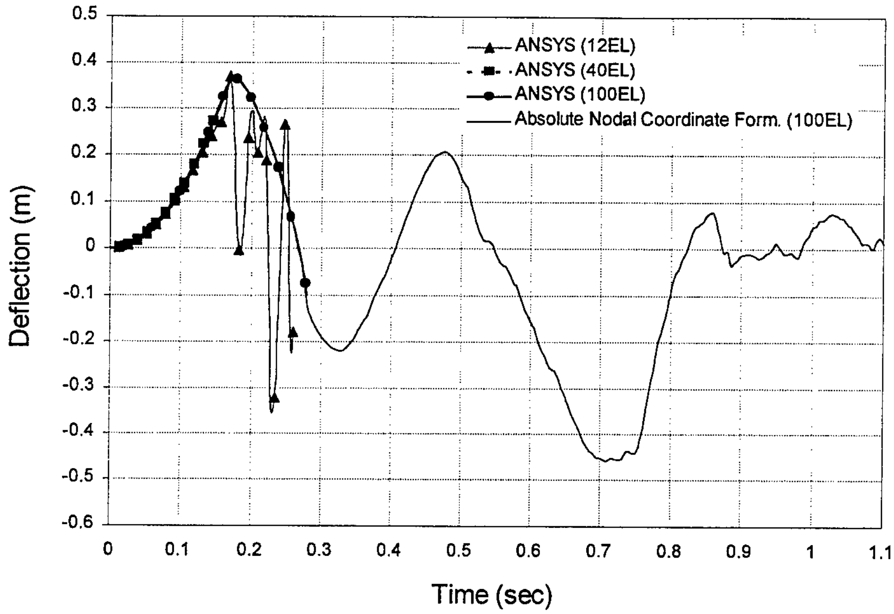


Fig. 14. Transverse deflection of the midpoint of the pendulum for different models. (a=50 m/s²)

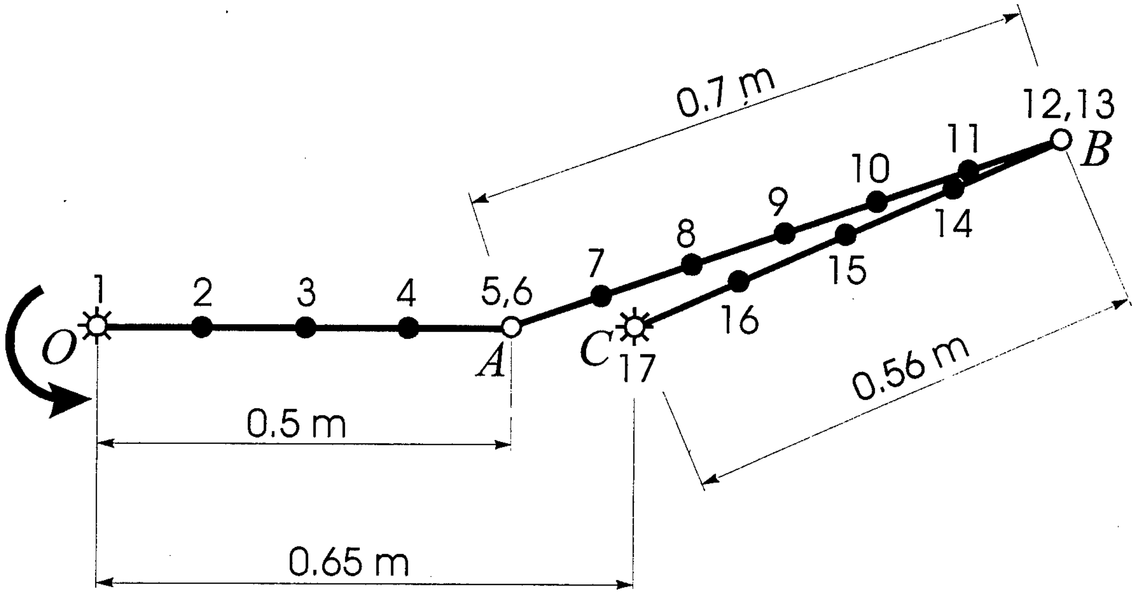


Fig. 15. The four bar mechanism in the original configuration

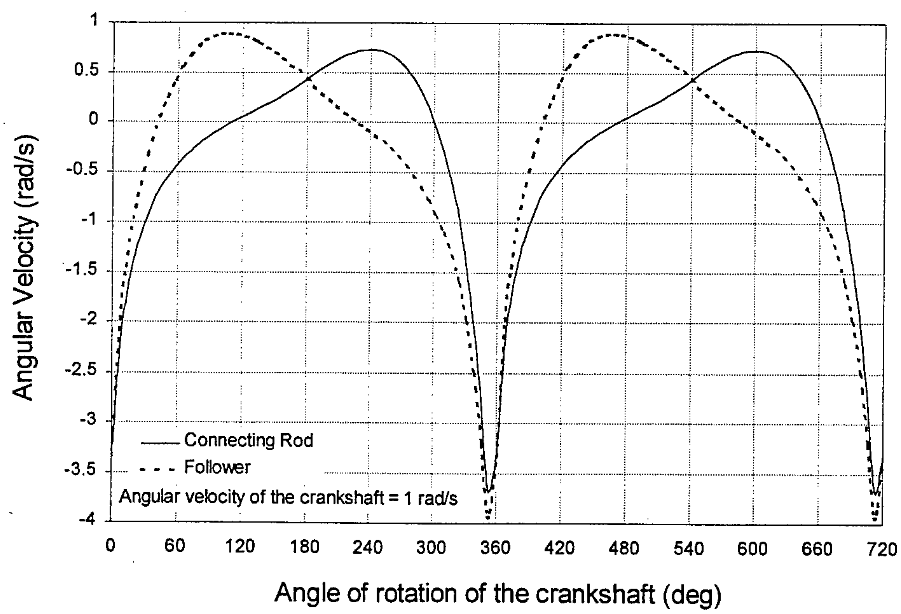


Fig. 16. Angular velocities of the connecting rod and the follower in the case of rigid body motion

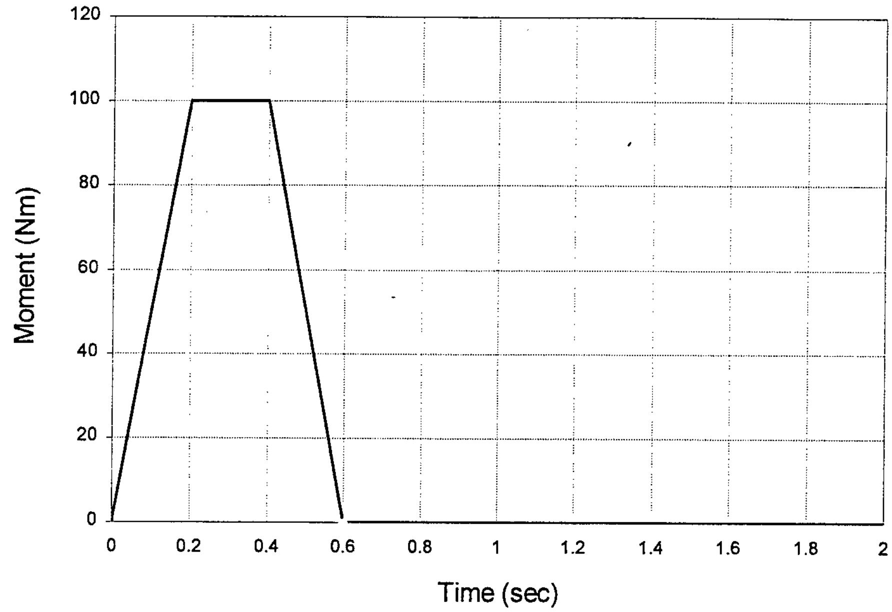


Fig. 17. Moment applied to the crankshaft

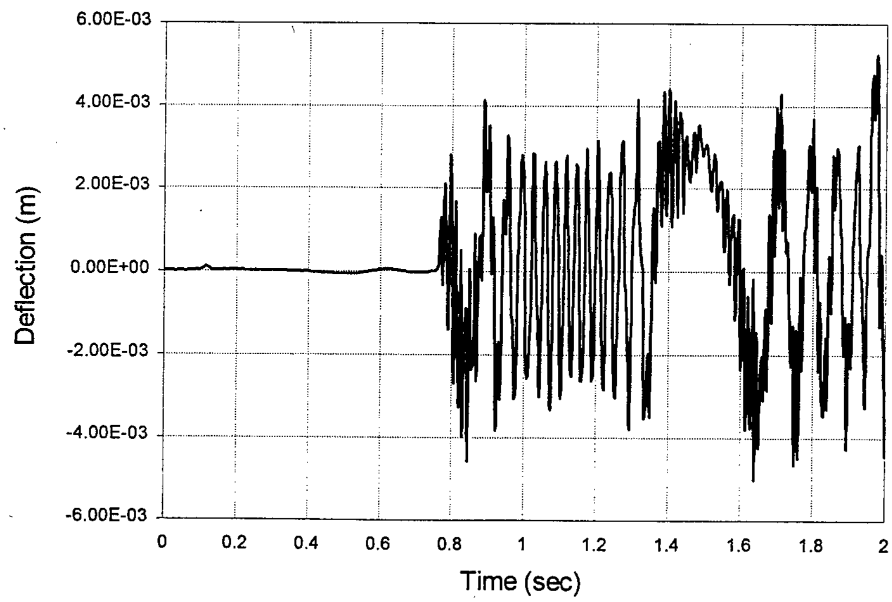


Fig. 18. Transverse deflection of the midpoint of the connecting rod obtained using the absolute nodal coordinate formulation

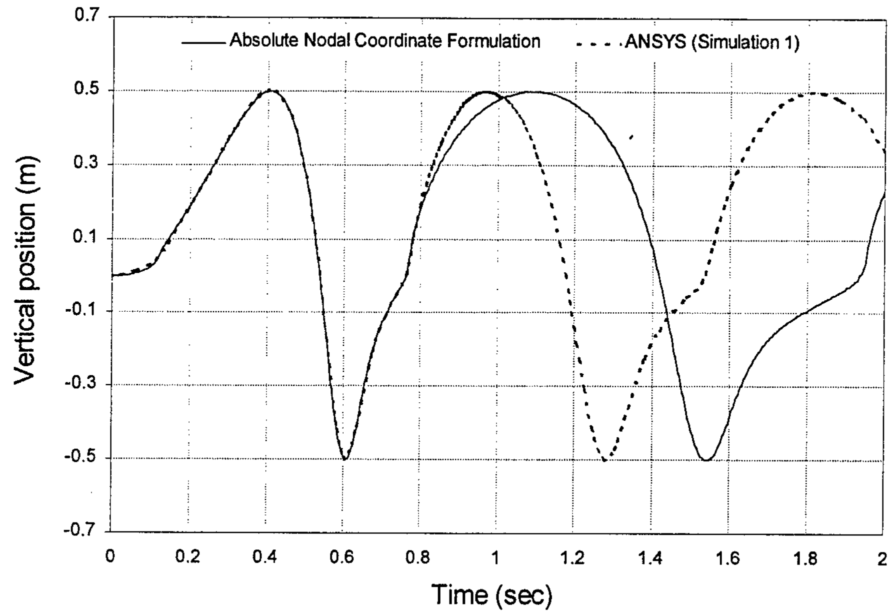


Fig. 19. Global vertical position of point *A* on the crankshaft

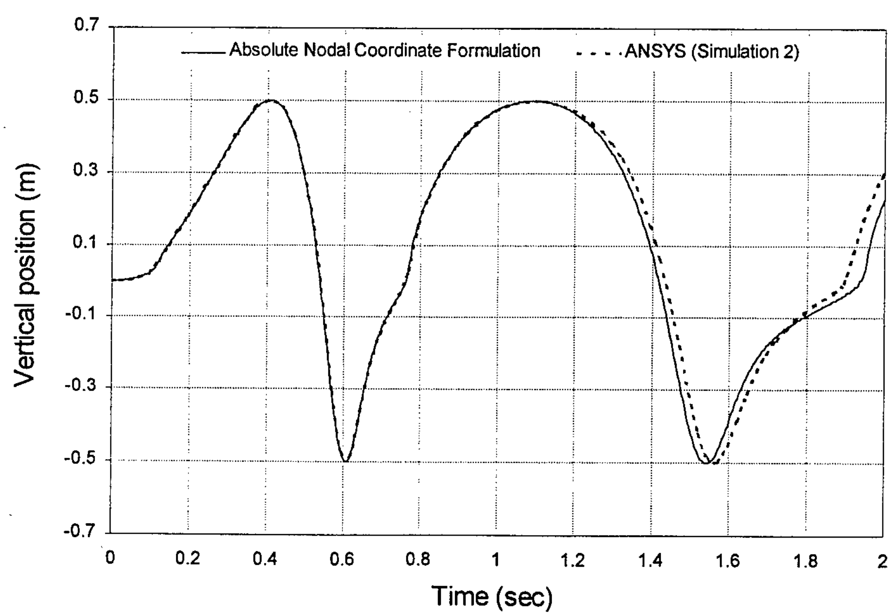


Fig. 20. Global vertical position of point *A* on the crankshaft

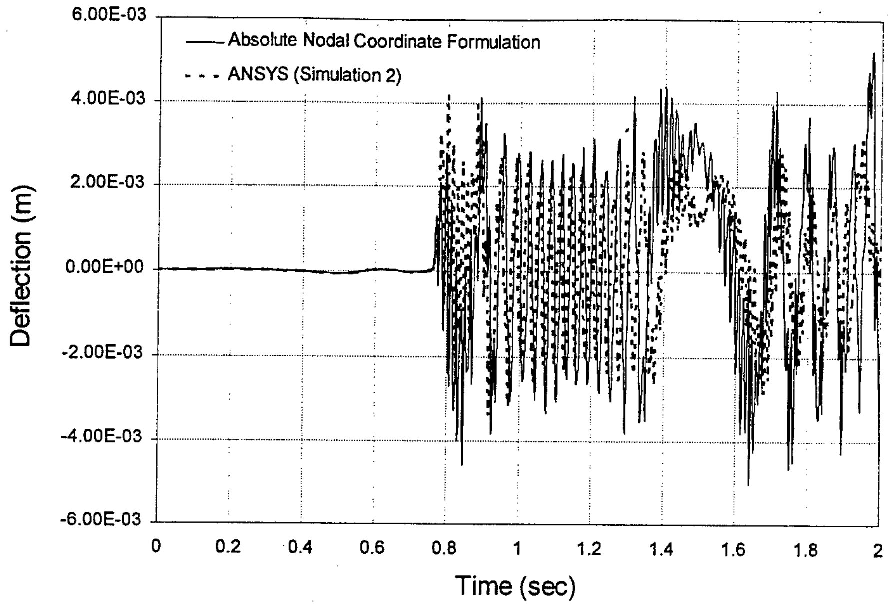


Fig. 21. Transverse deflection of the midpoint of the connecting rod

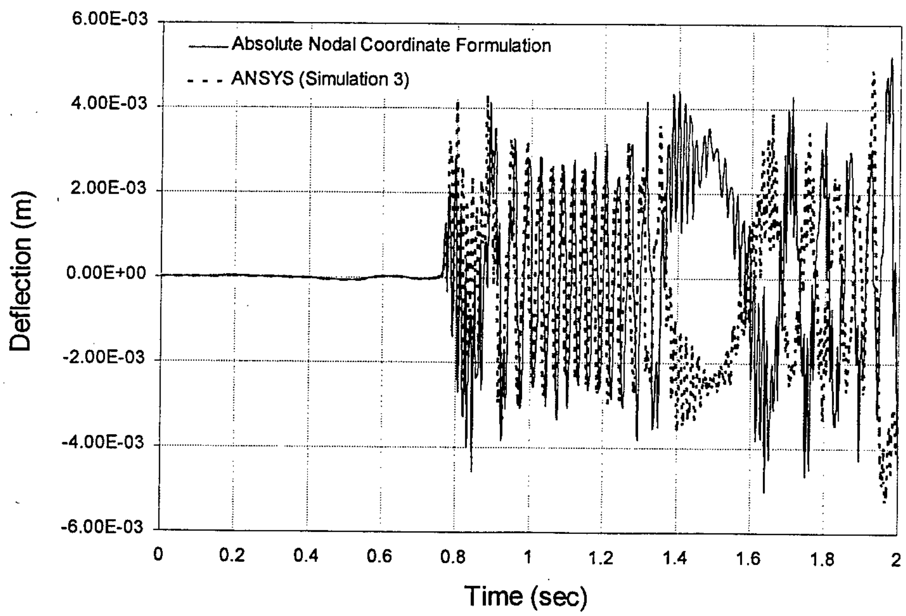


Fig. 22. Transverse deflection of the midpoint of the connecting rod